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Composition of quantum states and dynamical subadditivity

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Abstract

We introduce a composition of quantum states of a bipartite system which is based on the reshuffling of density matrices. This non-Abelian product is associative and stems from the composition of quantum maps acting on a simple quantum system. It induces a semi-group in the subset of states with maximally mixed partial traces. Subadditivity of the von Neumann entropy with respect to this product is proved. It is equivalent to subadditivity of the entropy of bistochastic maps with respect to their composition, where the entropy of a map is the entropy of the corresponding state under the Jamiołkowski isomorphism. Strong dynamical subadditivity of a concatenation of three bistochastic maps is established. Analogous bounds for the entropy of a composition are derived for general stochastic maps. In the classical case they lead to new bounds for the entropy of a product of two stochastic matrices.

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1. Introduction

General quantum dynamics are described by quantum stochastic maps, also called quantum channels or quantum operations. It is therefore crucial to investigate their properties in order to understand admissible dynamics in a quantum state space. The Jamiołkowski isomorphism [7, 18] associates with any quantum stochastic operation a quantum state acting on an extended space. Up to normalization, this state is nothing else than the dynamical or Choi matrix of the map. Thus the features of a quantum map are encoded in a state.

The spectral decomposition of the Jamiołkowski state yields a canonical Kraus decomposition of the map, i.e. realizes the map in terms of measurement operators. For a unitary evolution, the corresponding state is pure and the Kraus form consists of a single unitary operator. For the completely depolarizing channel, the state is maximally mixed and

the Kraus decomposition consists of many terms. The degree of mixing of the measurement operators required to construct a quantum map by its canonical Kraus form can therefore be estimated by a quantity like the entropy of the map which is actually the entropy of the associated state. Thus this entropy vanishes for a unitary evolution and reaches its maximal value for the completely depolarizing channel.

The aim of this paper is to analyse properties of the composition of stochastic maps [12]. Using the subadditivity of entropy for composite systems we prove an analogous dynamical subadditivity for bistochastic maps, i.e. maps which preserve the identity, see (52). A similar inequality proved for the concatenation of three bistochastic maps may be called strong dynamical subadditivity (69). Dynamical subadditivity generalizes to general stochastic maps by adding an extra term which vanishes for bistochastic maps. Restricting to diagonal states we obtain bounds for the entropy of the product of two classical stochastic matrices, see (88). This generalizes a recent result of Słomczyński on entropy of a product of bistochastic matrices [15].

Composition of quantum maps induces an action in the space of quantum states on a bipartite system. We analyse properties of this action which allows us to construct a semi-group in the space of Hermitian matrices. Composition of states induces also a semi-group structure in the set of positive definite operators on a bipartite system whose partial traces are proportional to the identity.

The main tools are coupling techniques, which associate with quantum maps states on composite systems, combined with subadditivity and strong subadditivity of quantum entropy. The explicit constructions that we use are related to similar constructions of Lindblad and techniques used in studying quantum dynamical entropy in the sense of [1]. There are also connections with quantum coherent information and inequalities as the quantum information data processing inequality, see [14].

The paper is organized as follows. In section 2 the necessary properties of quantum states and quantum maps are reviewed. In particular we consider the case of quantum operations with a diagonal dynamical matrix and show that any such stochastic or bistochastic quantum map reduces to a stochastic or bistochastic matrix acting on classical probability vectors. The notion of composition of states is introduced in section 3 where some of its properties are analysed. In section 4 we analyse the entropy of maps and formulate dynamical and strong dynamical subadditivity for compositions of bistochastic maps. Furthermore we discuss analogous results in a more general case of stochastic maps.

2. Quantum states and quantum maps

2.1. Quantum dynamical matrices

Let ρ denote an N -dimensional density matrix, i.e. a Hermitian, positive operator, satisfying the trace normalization condition $\text{Tr } \rho = 1$. The expectation value of an observable X , i.e. an N -dimensional matrix, is given by the usual relation

$$\langle X \rangle = \text{Tr } \rho X. \quad (1)$$

Such expectation functionals are called states. We shall in what follows identify $\langle \cdot \rangle$ with its corresponding ρ . Let \mathcal{D}_N denote the set of generally mixed quantum states acting on the N -dimensional Hilbert space \mathcal{H}_N . It is a convex, compact set of real dimensionality $N^2 - 1$. In the case of a qubit, i.e. $N = 2$, the space of mixed states is the Bloch ball, $\mathcal{D}_2 = B_3 \subset \mathbb{R}^3$. More generally, there is a one to one correspondence between arbitrary, i.e. not necessarily positive linear functionals F on the observables and N -dimensional matrices ρ

$$F(X) = \text{Tr } \rho X. \quad (2)$$

General quantum maps, sometimes called super-operators, are linear transformations either of the observables (Heisenberg picture) or of the functionals on the observables (Schrödinger picture). In this paper we shall mostly use the Schrödinger picture and denote such quantum maps by Φ . The adjoint Φ^\dagger of a super-operator Φ is given by

$$\Phi^\dagger(\rho) = (\Phi(\rho^*))^*, \quad (3)$$

where star denotes the Hermitian adjoint.

Fixing a basis in \mathcal{H}_N we identify an N -dimensional matrix ρ with a vector of dimension N^2 just by writing the entries of ρ in lexicographical order. So the entry $\rho_{m\mu}$ is placed on the $((m-1)N + \mu)$ th row. A general linear quantum map,

$$\rho \mapsto \rho' := \Phi(\rho), \quad (4)$$

may be described by a matrix of size N^2 still denoted by Φ ,

$$\rho'_{m\mu} = \Phi_{nv}^{m\mu} \rho_{nv}, \quad (5)$$

where Einstein's summation convention is taken. We use the following notation for matrix elements of an operator acting on a composed Hilbert space and represented in a product basis,

$$\Phi_{nv}^{m\mu} := \langle m \otimes \mu | \Phi | n \otimes v \rangle. \quad (6)$$

Another convenient way to describe quantum maps is to use the Choi-Jamiołkowski encoding, also called dynamical matrix or Choi matrix D_Φ [16]. It amounts to a reordering of matrix elements of Φ

$$D_\Phi \equiv \Phi^R \quad \text{so that} \quad (D_\Phi)_{\mu\nu}^{mn} = (\Phi^R)_{\mu\nu}^{mn} = \Phi_{nv}^{m\mu}. \quad (7)$$

Let us consider the projector on the maximally entangled state $|\psi^+\rangle := \frac{1}{\sqrt{N}} \sum_{m=1}^N |m\rangle \otimes |m\rangle$

$$P_+ := |\psi^+\rangle\langle\psi^+| = \frac{1}{N} \sum_{m\mu} |m\rangle\langle\mu| \otimes |m\rangle\langle\mu|. \quad (8)$$

We can in a similar way as above identify P_+ with a vector in a space of dimension $N^2 \times N^2$. Its entries are

$$\begin{aligned} (P_+)_{mn\mu\nu} &= \frac{1}{N} \sum_{k\kappa} (|k\rangle\langle\kappa| \otimes |k\rangle\langle\kappa|)_{mn\mu\nu} = \frac{1}{N} \sum_{k\kappa} (|k\rangle\langle\kappa|)_{m\mu} (|k\rangle\langle\kappa|)_{n\nu} \\ &= \frac{1}{N} \sum_{k\kappa} \delta_{km} \delta_{\kappa\mu} \delta_{kn} \delta_{\kappa\nu} = \frac{1}{N} \delta_{mn} \delta_{\mu\nu}. \end{aligned} \quad (9)$$

We now compute

$$\begin{aligned} ((\Phi \otimes \text{id})(P_+))_{mn\mu\nu} &= (\Phi \otimes \text{id})_{m'n'\mu'\nu'}^{mn\mu\nu} (P_+)_{m'n'\mu'\nu'} \\ &= \Phi_{m'\mu'}^{m\mu} \delta_{nn'} \delta_{\nu\nu'} (P_+)_{m'n'\mu'\nu'} \\ &= \frac{1}{N} \Phi_{m'\mu'}^{m\mu} \delta_{nn'} \delta_{\nu\nu'} \delta_{m'n'} \delta_{\mu\nu'} \\ &= \frac{1}{N} \Phi_{nv}^{m\mu} = \frac{1}{N} (D_\Phi)_{\mu\nu}^{mn}. \end{aligned} \quad (10)$$

So we see that, up to a factor N , D_Φ is the action of $\text{id} \otimes \Phi$ on the one-dimensional projection P_+ on the maximally entangled state. The map $\Phi \mapsto D_\Phi$ is linear and it intertwines adjoints

$$D_{\Phi^\dagger} = (D_\Phi)^*. \quad (11)$$

Equation (7) may be considered as the definition of the reshuffling transformation, written $\Phi \rightarrow \Phi^R$, which is defined for any matrix Φ acting on the Hilbert space $\mathcal{H}_{N^2} = \mathcal{H}_N \otimes \mathcal{H}_N$

[18]. It should be stressed that the reshuffling operation depends on the distinguished basis in \mathcal{H}_N that we have used.

Choi's theorem [4], proves that a map Φ is completely positive (CP), which means that the extended map $\Phi \otimes \text{id}$ is positive for any size of the extension, if and only if the corresponding dynamical matrix D_Φ is positive, $D_\Phi \geq 0$. The eigenvalue decomposition of the dynamical matrix of a CP map Φ leads to the canonical Kraus form [8] of the map

$$\Phi(\rho) = \sum_{\alpha=1}^{N^2} A_\alpha \rho A_\alpha^\dagger, \quad (12)$$

where the Kraus operators are orthogonal

$$\langle A_\alpha | A_\beta \rangle = \text{Tr } A_\alpha^\dagger A_\beta = d_\alpha \delta_{\alpha\beta}, \quad (13)$$

so that the non-negative weights d_α become the eigenvalues of the dynamical matrix D_Φ . Hence, in this (almost) canonical form the number of Kraus operators does not exceed N^2 .

A quantum map Φ is trace preserving (TP) if $\text{Tr } \Phi(\rho) = \text{Tr } \rho$ for any ρ . The corresponding dynamical matrix D_Φ acts on the composite Hilbert space $\mathcal{H}_{N^2} = \mathcal{H}_A \otimes \mathcal{H}_B$ and, in terms of the dynamical matrix, trace preserving means that

$$\text{Tr}_A D_\Phi = \mathbb{1}. \quad (14)$$

This implies in particular that $\text{Tr } D_\Phi = N$. Completely positive trace-preserving maps (CPTP maps) are often called quantum operations or quantum stochastic maps. Since the dynamical map of a quantum operation Φ is positive and normalized as in (14) the rescaled matrix $\frac{1}{N} D_\Phi$ is a state on the extended Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$, see [7, 3]. We shall say that

$$\zeta := \frac{1}{N} D_\Phi \quad (15)$$

is the Jamiołkowski state associated with Φ . A quantum map is called unital if it leaves the maximally mixed state invariant. This is the case iff

$$\text{Tr}_B D_\Phi = \mathbb{1}, \quad (16)$$

a condition dual to (14). A CP quantum map which is trace preserving and unital is called bistochastic. A composition of bistochastic maps is still bistochastic.

2.2. Classical case—diagonal dynamical matrices

Let us diagonalize a density matrix. The elements on the diagonal may be interpreted as a classical probability vector P with components $p_i = \rho_{ii}$, i.e. $p_i \geq 0$ and $\sum_{i=1}^N p_i = 1$. In a similar way a quantum map Φ reduces to a classical one if the dynamical matrix $D = \Phi^R$ is diagonal, $D_{ab} = T_{ab} \delta_{ac} \delta_{bd}$.

Reshaping the diagonal of the dynamical matrix which has dimension N^2 one obtains a matrix T of dimension N

$$T = T(\Phi) \quad \text{with} \quad T_{ij} = \Phi_{jj}^{ii}, \quad (17)$$

with no summation performed. Positivity of D implies that all elements of T are non-negative. Furthermore, the partial trace condition (14) implies that the matrix T is stochastic since $\sum_{i=1}^N T_{ij} = 1$ for all $j = 1, 2, \dots, N$. In fact, the action of a diagonal dynamical matrix D on diagonal density matrices reduces to a Markov transition of a probability vector, $P' = TP$.

If, additionally, the complementary partial trace condition (16) holds, then the matrix T is bistochastic $\sum_{j=1}^N T_{ij} = 1$ for all $i = 1, 2, \dots, N$ and the uniform vector $P_* := (\frac{1}{N}, \dots, \frac{1}{N})$

is invariant under multiplication by a bistochastic matrix T . Hence quantum stochastic and bistochastic maps acting in the space \mathcal{D}_N of quantum states can be considered as a direct generalizations of stochastic and bistochastic matrices, which act on classical probability vectors.

Note that the quantum identity map, $\Phi = \text{id}$, is not classical, since the dynamical matrix $D = \Phi^R$ is not diagonal, and the off-diagonal elements of ρ are preserved. On the other hand, the coarse-graining map,

$$\Phi_{\text{CG}}(\rho) := \sum_{i=1}^N |i\rangle\langle i|\rho|i\rangle\langle i| \tag{18}$$

which strips away all off-diagonal elements of a density operator is described by a diagonal dynamical matrix. The corresponding stochastic matrix is the identity, $T(\Phi_{\text{CG}}) = \mathbb{1}$, since all diagonal elements of ρ remain untouched under the action of (18). Let us also distinguish the flat stochastic matrix T_* whose elements are all equal, $(T_*)_{ij} = \frac{1}{N}$. It is a bistochastic matrix which maps any probability vector P into the uniform one, $T_*P = P_*$.

2.3. Entropies of maps and states

To any normalized probability vector P of size N we may associate its Shannon entropy

$$H(P) := - \sum_{i=1}^N p_i \ln p_i = \sum_{i=1}^N \eta(p_i), \tag{19}$$

where we have introduced the function

$$\eta(x) := -x \ln x \quad \text{for } x > 0 \quad \text{and} \quad \eta(0) := 0. \tag{20}$$

This entropy is a measure for the mixedness of a probability vector. In a similar way the degree of mixing of a quantum state ρ is characterized by its von Neumann entropy

$$S(\rho) := -\text{Tr} \rho \ln \rho = \text{Tr} \eta(\rho), \tag{21}$$

equal to the Shannon entropy of its spectrum. The entropy varies from zero for a pure state to $\ln N$ for the maximally mixed state, $\rho_* = \frac{1}{N}\mathbb{1}$.

The density matrix $\zeta = D_\Phi/N$ associated with a quantum stochastic map Φ depends on the basis that is used to compute the entries of Φ , see (5). A change of basis in \mathcal{H}_N corresponds to a unitary transformation of D_Φ , therefore the eigenvalues of D_Φ do not change. It is then natural to consider the von Neumann entropy of the bipartite state associated with Φ by the Jamiolkowski isomorphism.

Definition 1. Let Φ be a trace-preserving completely positive map with associated Jamiolkowski state $\zeta = \frac{1}{N}D_\Phi$. The entropy of the quantum operation Φ is defined to be

$$S(\Phi) := S(\zeta) = S\left(\frac{1}{N}D_\Phi\right). \tag{22}$$

Since the state $\zeta = D_\Phi/N$ acts on the extended Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$ the entropy of the map varies from zero for a unitary dynamics to $2 \ln N$ for a completely depolarizing channel Φ_* , which sends any state to the maximally mixed state, $\Phi_*(\rho) = \rho_*$, see [18].

Let us now move to a classical discrete dynamics in the probability simplex. The following definition of entropy of a stochastic matrix introduced in [15, 19]

$$H_I(T) := \sum_{j=1}^N p_j^I H(t_j), \tag{23}$$

is decorated by a label ‘I’, as it is based on an invariant state of a matrix, $P^I = T P^I$. Here t_j denotes the j th column of a transition matrix T , so (23) represents the average Shannon entropy of columns of T weighted by its invariant state P^I .

To demonstrate a direct relation to the quantum dynamics we shall use a simplified version of the entropy of a transition matrix

$$H(T) := -\frac{1}{N} \sum_{i=1}^N \sum_{j=1}^N T_{ij} \ln T_{ji}. \quad (24)$$

Observe that for any bistochastic matrix the uniform vector is invariant, $P^I = P_*$, so $p_j^I = \frac{1}{N}$ and both definitions of entropy do coincide. Both quantities, $H_1(T)$ and $H(T)$, vary from zero to $\ln N$.

Using equation (17) one concludes that for any stochastic map Φ represented by a diagonal dynamical matrix D_Φ its entropy is up to a constant equal to the entropy of the associated stochastic matrix,

$$S(\Phi) = H(T(\Phi)) + \ln N. \quad (25)$$

The constant $\ln N$ is due to the $\frac{1}{N}$ normalization factor in front of the dynamical matrix, see definition (22). Although in general the entropy of a quantum map belongs to $[0, 2 \ln N]$, the entropy of the maps represented by diagonal D and corresponding to stochastic matrices varies from $\ln N$ to $2 \ln N$. Among this class the minimal entropy characterizes the coarse-graining map (18), for which $T(\Phi_{CG}) = \mathbb{1}$, and $S(\Phi_{CG}) = \ln N$. The maximum is achieved for the completely depolarizing channel, Φ_* , since $T(\Phi_*) = T_*$ so that $S(\Phi_*) = H(T_*) + \ln N = 2 \ln N$.

2.4. Entropy exchange and Lindblad’s theorem

Consider a CP map Φ represented in its canonical Kraus form (12). For any state $\rho \in \mathcal{D}_N$ define a positive operator $\hat{\sigma} = \hat{\sigma}(\Phi, \rho)$ acting on the extended Hilbert space \mathcal{H}_{N^2} ,

$$\hat{\sigma}_{\alpha\beta} := \text{Tr} \rho A_\beta^\dagger A_\alpha, \quad \alpha, \beta = 1, \dots, N^2. \quad (26)$$

If the map Φ is stochastic, then the operator $\hat{\sigma}$ is also normalized in the sense that

$$\text{Tr} \hat{\sigma} = \sum_{\alpha=1}^{N^2} \text{Tr} \rho A_\alpha^\dagger A_\alpha = \text{Tr} \rho = 1 \quad (27)$$

and so it represents a density operator in its own right, $\hat{\sigma} \in \mathcal{D}_{N^2}$. In particular, if $\rho = \rho_* = \frac{1}{N} \mathbb{1}$ then, using the canonical Kraus decomposition (12), one shows that

$$\hat{\sigma}(\Phi, \rho_*) = \frac{1}{N} D_\Phi = \varsigma. \quad (28)$$

The von Neumann entropy of $\hat{\sigma}$ depends on ρ , and equals $S(\Phi)$, as defined above, if ρ is the maximally mixed state.

Auxiliary states $\hat{\sigma}$ in an extended Hilbert space were used by Lindblad to derive bounds for the entropy of the image $\rho' = \Phi(\rho)$ of an initial state under the action of a CPTP map. Lindblad’s bounds [10]

$$|S(\hat{\sigma}) - S(\rho)| \leq S(\rho') \leq S(\hat{\sigma}) + S(\rho) \quad (29)$$

are obtained by defining yet another density matrix in the composite Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_M$:

$$\omega := \sum_{\alpha=1}^M \sum_{\beta=1}^M A_\alpha \rho A_\beta^\dagger \otimes |\alpha\rangle\langle\beta|, \quad (30)$$

where $M = N^2$ and $\{|\alpha\rangle\}$ is an orthonormal basis in \mathcal{H}_M . Computing partial traces one finds that

$$\text{Tr}_N \omega = \hat{\sigma} \quad \text{and} \quad \text{Tr}_M \omega = \rho'. \quad (31)$$

It is possible to verify that $S(\omega) = S(\rho)$, and so one arrives at (29) using subadditivity of the entropy and the triangle inequality [2]. These results can be obtained using the first part of the proof of theorem 1.

If the initial state is pure, that is if $S(\rho) = 0$, we find that the final state ρ' has entropy $S(\hat{\sigma})$. For this reason $S(\hat{\sigma})$ was called the entropy exchange of the operation Φ by Shumacher [13]. In that work an alternative representation of the entropy exchange was given

$$S(\hat{\sigma}(\Phi, \rho)) = S((\text{id} \otimes \Phi)|\varphi\rangle\langle\varphi|), \quad (32)$$

where $|\varphi\rangle$ is an arbitrary purification of the mixed state, $\text{Tr}_B |\varphi\rangle\langle\varphi| = \rho$. To prove this useful relation it is enough to find a pure state in an extended Hilbert space, such that one of its partial traces gives $\hat{\sigma}$ and the other one the argument of the entropy function at the right-hand side of (32).

A kind of classical analogue of the quantum entropy bound (29) of Lindblad was proved later by Słomczyński [15]. He introduced the notion of entropy of a stochastic matrix T with respect to an arbitrary probability distribution $P = \{p_i\}_{i=1}^N$,

$$H_P(T) := \sum_{i=1}^N p_i H(\vec{t}_i) \quad \vec{t}_i = (T_{1i}, T_{2i}, \dots, T_{Ni}). \quad (33)$$

This quantity—an average entropy of columns of matrix T weighted by probability vector P —allows us to obtain the bounds for the entropy of a classically transformed state, $P' = TP$,

$$H_P(T) \leq H(P') \leq H_P(T) + H(P). \quad (34)$$

These bounds look somewhat similar to the quantum result (29) of Lindblad, but a careful comparison is required. Applying the definition (26) of the auxiliary state $\hat{\sigma}$ to the classical case of a diagonal state, $\rho_{ii} = p_i \delta_{ij}$ and a diagonal dynamical matrix D we find

$$S(\sigma^{\text{class}}) = H_P(T) + H(P), \quad (35)$$

where $T = T(D^R)$ is the classical stochastic matrix given by (17). Substituting this result into the Lindblad bound (29), and renaming ρ and ρ' into P and P' we realize that the argument of the absolute value in the lower bound reduces to $H_P(T)$ and is not negative, so we arrive at

$$H_P(T) \leq H(P') \leq H_P(T) + 2H(P). \quad (36)$$

The lower bound coincides exactly with the result (34) of Słomczyński. The upper bound is weaker (note the presence of the term $2H(P)$ instead of $H(P)$), but it holds in general for all quantum maps, while (34) is true for classical dynamics only.

3. Composition of maps and composition of states

We first recall some properties of the reshuffling transformation of a matrix as defined in (7). Reshuffling does not preserve the spectrum nor the Hermiticity of a matrix. It is an involution, since performing this transformation twice returns the initial matrix, $(X^R)^R = X$.

Using the Jamiołkowski isomorphism the composition of maps acting on a single quantum system can be used to define a composition between quantum states of a bi-partite system, in fact, this composition extends to arbitrary matrices.

Definition 2. The reshuffling operation (7) defines a composition between arbitrary matrices σ_1 and σ_2 on a composite system $\mathcal{H}_N \otimes \mathcal{H}_N$

$$\sigma_1 \odot \sigma_2 := (\sigma_1^R \sigma_2^R)^R. \quad (37)$$

For stochastic matrices obtained by reshaping two diagonal density matrices $T_1 = T(\sigma_1^R)$ and $T_2 = T(\sigma_2^R)$ according to (17), the composition of the diagonal states returns the usual multiplication of stochastic matrices,

$$T((\sigma_1 \odot \sigma_2)^R) = T(\sigma_1^R)T(\sigma_2^R) = T_1 T_2. \quad (38)$$

Using the definition of reshuffling we see that generally $(D^R)^2$ differs from $(D^2)^R$. Therefore the composition performed on two copies of a state σ differs from its square

$$\sigma^{\odot 2} := \sigma \odot \sigma \neq \sigma^2. \quad (39)$$

3.1. Properties of the composition

Lemma 1. Let X and Y denote two matrices of size N^2 .

$$\text{If } X \geq 0 \text{ and } Y \geq 0 \text{ then } X \odot Y = (X^R Y^R)^R \geq 0. \quad (40)$$

Proof. Denoting by Ψ and Φ the completely positive maps with corresponding dynamical matrices X and Y we see that $X \odot Y$ is the dynamical matrix of the composed map $\Psi \circ \Phi$. Since the composition of two completely positive maps yields again a completely positive map [3], we infer (40). \square

Proposition 1. The set of all operators acting on a composite Hilbert space $\mathcal{H}_N \otimes \mathcal{H}_N$, equipped with the composition law (\odot) , is a non-Abelian associative semi-group. Moreover, if σ_1 and σ_2 are Hermitian operators on $\mathcal{H}_N \otimes \mathcal{H}_N$ then also $\sigma_1 \odot \sigma_2$ is Hermitian. Therefore the set of all Hermitian operators on $\mathcal{H}_N \otimes \mathcal{H}_N$ is a non-Abelian associative subsemi-group.

Proof. Let P_+ denote the projector on the maximally entangled state. Since $(P_+)^R = \mathbb{1}$, this operator plays the role of the neutral element of the composition. The composition \odot is non-Abelian, $\sigma_1 \odot \sigma_2 \neq \sigma_2 \odot \sigma_1$, because the composition of quantum maps is not commutative. It is on the other hand associative $(\sigma_1 \odot \sigma_2) \odot \sigma_3 = \sigma_1 \odot (\sigma_2 \odot \sigma_3)$, since the composition of maps is.

It remains to prove that $\sigma_1 \odot \sigma_2$ is Hermitian if σ_1 and σ_2 are. Let Φ be a super-operator associated with D_Φ through the Jamiołkowski isomorphism, then $(D_\Phi)^* = D_{\Phi^\dagger}$. Moreover, for the two super-operators Φ and Ψ we have

$$(\Phi \circ \Psi)^\dagger(X) = (\Phi \circ \Psi(X^*))^* = (\Phi(\Psi(X^*)))^* = \Phi^\dagger((\Psi(X^*))^*) = (\Phi^\dagger \circ \Psi^\dagger)(X). \quad (41)$$

Using

$$D_{\Phi \circ \Psi} = D_\Phi \odot D_\Psi \quad (42)$$

completes the proof. \square

Restricting our attention to the set \mathcal{D}_{N^2} of quantum states on a composite system, we see that this algebraic structure breaks down since the trace condition, $\text{Tr } \sigma = 1$, is not preserved under composition. However, one may overcome this difficulty by selecting a certain subset of quantum states. Thus consider the subset $\mathcal{D}_{N^2}^1$ of density matrices of a composite system

$\mathcal{H}_A \otimes \mathcal{H}_B$ of size N^2 such that their partial trace over the first system is the maximally mixed state

$$\mathcal{D}_{N^2}^I := \left\{ \sigma \in \mathcal{D}_{N^2} : \text{Tr}_A \sigma = \frac{1}{N} \mathbb{1} \right\}. \quad (43)$$

With respect to the Jamiołkowski isomorphism these states correspond to the trace-preserving maps. Since a composition of any two trace-preserving maps preserves the trace we infer that the composition (37) acts internally in the set $\mathcal{D}_{N^2}^I$.

Proposition 2. *The set of all operators acting on the Hilbert space \mathcal{H}_N of a bipartite system such that their left marginal is proportional to the identity, equipped with the composition \odot is a non-Abelian associative semi-group.*

It is convenient to distinguish another composition sub-algebra by defining the set of states with both marginals proportional to the identity,

$$\mathcal{D}_{N^2}^{II} := \left\{ \sigma \in \mathcal{D}_{N^2}^I : \text{Tr}_B \sigma = \frac{1}{N} \mathbb{1} \right\}. \quad (44)$$

Due to condition (16) this semi-group is generated by compositions of bistochastic maps.

3.2. Idempotent states

Consider the state σ of a bipartite system obtained by extending an arbitrary state ρ by the maximally mixed state

$$\sigma := \frac{1}{N} \rho \otimes \mathbb{1}. \quad (45)$$

This state is proportional to the dynamical matrix D of the operation Φ_ρ , which acts as a complete single-step contraction, sending any initial state ω into ρ ,

$$\sigma = \frac{1}{N} D_{\Phi_\rho}, \quad \text{where} \quad \Phi_\rho(\omega) := \rho \quad (46)$$

for any $\omega \in \mathcal{D}_N$. To show this let us start with the dynamical matrix of this map, $D_{\mu\nu}^{mn} = \rho_{m\mu} \delta_{nv}$. Writing out the matrix entries of

$$\omega' := \Phi_\rho(\omega) = D^R \omega = (\rho \otimes \mathbb{1})^R \omega \quad (47)$$

in the standard basis we obtain the desired result

$$\omega'_{m\mu} = D_{\mu\nu}^{mn} \omega_{nv} = \rho_{m\mu} (\text{Tr } \omega) = \rho_{m\mu}. \quad (48)$$

A state σ of an algebra is called idempotent if $\sigma \odot \sigma = \sigma$. Hence any state of the form (45) is idempotent, since the single-step contracting map Φ_ρ applied for the second time does no longer influence the system, $\Phi_\rho \circ \Phi_\rho = \Phi_\rho$.

4. Entropy of a composition

In this section, we analyse the behaviour of the entropy of a quantum operation under composition. The bounds (34) on the increase of entropy of probability vectors under discrete dynamics allowed Słomczyński to prove the subadditivity relation [15, 19]

$$H_I(T_1) \leq H_I(T_2 T_1) \leq H_I(T_1) + H_I(T_2), \quad (49)$$

provided both stochastic matrices T_1 and T_2 have the same invariant state, $P_1^I = P_2^I$. Restricting our attention to the case of bistochastic matrices for which $P_1^I = P_2^I = P_* = \{\frac{1}{N}, \dots, \frac{1}{N}\}$ we use (24) instead of (23) and drop the label 'I' in the subadditivity relation to get

$$H(T_1) \leq H(T_2 T_1) \leq H(T_1) + H(T_2). \quad (50)$$

Considering a product of three bistochastic matrices Słomczyński proved [15] a strong subadditivity relation for classical dynamics,

$$H(T_3 T_2 T_1) + H(T_2) \leq H(T_3 T_2) + H(T_2 T_1). \quad (51)$$

4.1. Dynamical subadditivity and strong subadditivity for bistochastic maps

Motivated by the classical results above we formulate and prove their quantum counterparts.

Theorem 1 (Dynamical subadditivity for bistochastic quantum operations). *Let Φ_1 be a bistochastic quantum operation and Φ_2 a general stochastic quantum map then their entropies satisfy the subadditivity inequality*

$$S(\Phi_2 \circ \Phi_1) \leq S(\Phi_1) + S(\Phi_2). \quad (52)$$

If both Φ_1 and Φ_2 are bistochastic then

$$\max(\{S(\Phi_1), S(\Phi_2)\}) \leq \min(\{S(\Phi_1 \circ \Phi_2), S(\Phi_2 \circ \Phi_1)\}). \quad (53)$$

An equivalent statement of (52) and (53) is the triangle inequality for composition

$$\max(\{S(\sigma_1), S(\sigma_2)\}) \leq S(\sigma_1 \odot \sigma_2) \leq S(\sigma_1) + S(\sigma_2), \quad (54)$$

where $\sigma_1, \sigma_2 \in \mathcal{D}_{N^2}^{\text{II}}$ and the set $\mathcal{D}_{N^2}^{\text{II}}$ has been defined in (44), and the same bounds hold for $S(\sigma_2 \odot \sigma_1)$.

Proof. (i) We first show the upper bound (52).

Let Θ be a stochastic quantum operation, generally not bistochastic, with Kraus form

$$\Theta(\rho) = \sum_{\alpha=1}^K C_{\alpha} \rho C_{\alpha}^{\dagger}. \quad (55)$$

Introduce a map from \mathcal{H}_N to $\mathcal{H}_N \otimes \mathcal{H}_N \otimes \mathcal{H}_K$ as follows: fix an orthonormal basis $\{|\alpha\rangle\}$ in \mathcal{H}_K and let

$$F\varphi := \sum_{\alpha=1}^K (C_{\alpha}\varphi) \otimes |\alpha\rangle, \quad (56)$$

The adjoint map acts as

$$F^{\dagger}\varphi \otimes |\alpha\rangle = C_{\alpha}^{\dagger}\varphi \quad (57)$$

and one checks that

$$F^{\dagger}F\varphi = F^{\dagger} \sum_{\beta=1}^K (C_{\beta}\varphi) \otimes |\beta\rangle = \sum_{\beta=1}^K C_{\beta}^{\dagger}C_{\beta}\varphi = \varphi, \quad (58)$$

since Θ is trace preserving. Therefore F is an isometry. It follows in particular that, for an arbitrary N -dimensional matrix ρ , $F\rho F^{\dagger}$ and ρ have up to multiplicities of zero the same eigenvalues.

Using F we can express the Lindblad operator ω , see (30), as

$$\omega = F\rho F^\dagger = \sum_{\alpha,\beta=1}^K C_\alpha \rho C_\beta^\dagger \otimes |\alpha\rangle\langle\beta|. \quad (59)$$

The operator ω is a density matrix on the composite system $\mathcal{H}_N \otimes \mathcal{H}_K$. If the initial state is maximally mixed, $\rho = \rho_*$, taking the partial trace over the first subsystem we obtain the following density matrix of the ancilla $E = \mathcal{H}_K$,

$$\rho = \text{Tr}_N \omega = \hat{\sigma}(\Theta, \rho_*) = \varsigma = \frac{1}{N} D_\Theta. \quad (60)$$

We perform the construction of above for the composition $\Phi_2 \circ \Phi_1$. We first write both quantum operations Φ_1 and Φ_2 in their Kraus forms

$$\Phi_1(\rho) = \sum_{\alpha=1}^M A_\alpha \rho A_\alpha^\dagger \quad \text{and} \quad \Phi_2(\rho) = \sum_{\alpha=1}^M B_\alpha \rho B_\alpha^\dagger \quad (61)$$

where $M = N^2$. Putting $\Theta = \Phi_2 \circ \Phi_1$ we consider its Kraus decomposition

$$C_{\alpha_2 \alpha_1} := B_{\alpha_2} A_{\alpha_1} \quad (62)$$

implying that $K = N^2 \times N^2 = N^4$. Consider a state ω_{21} acting on an extended tripartite space $\mathcal{H}_N \otimes \mathcal{H}_M \otimes \mathcal{H}_M = \mathcal{H}_N \otimes E_2 \otimes E_1$

$$\begin{aligned} \omega_{21} &= \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2=1}^N C_{\alpha_2 \alpha_1} \rho C_{\beta_2 \beta_1}^\dagger \otimes |\alpha_2 \otimes \alpha_1\rangle\langle\beta_2 \otimes \beta_1| \\ &= \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2=1}^N B_{\alpha_2} A_{\alpha_1} \rho (B_{\beta_2} A_{\beta_1})^\dagger \otimes |\alpha_2 \otimes \alpha_1\rangle\langle\beta_2 \otimes \beta_1| \end{aligned} \quad (63)$$

and let ρ_{21} be the restriction of ω_{21} to the ancilla $E_2 E_1 = \mathcal{H}_M \otimes \mathcal{H}_M$. We also compute the restrictions ρ_1 and ρ_2 to the first and second ancilla. Assuming that $\rho = \rho_* = \frac{1}{N} \mathbb{1}$ we obtain

$$\rho_{12} = \frac{1}{N} D_{\Phi_2 \circ \Phi_1} \quad (64)$$

$$\rho_1 = \text{Tr}_{\mathcal{H}_N \otimes E_2} \omega_{21} = \text{Tr}_{\mathcal{H}_N} \frac{1}{N} \sum_{\alpha_1, \beta_1=1}^N A_{\alpha_1} \mathbb{1} A_{\beta_1}^\dagger \otimes |\alpha_1\rangle\langle\beta_1| = \frac{1}{N} D_{\Phi_1} \quad (65)$$

$$\rho_2 = \text{Tr}_{\mathcal{H}_N \otimes E_1} \omega_{21} = \text{Tr}_{\mathcal{H}_N} \frac{1}{N} \sum_{\alpha_2, \beta_2=1}^N B_{\alpha_2} \mathbb{1} B_{\beta_2}^\dagger \otimes |\alpha_2\rangle\langle\beta_2| = \frac{1}{N} D_{\Phi_2}. \quad (66)$$

The last inequality holds because Φ_1 is bistochastic. The upper bound of (52) now follows by applying subadditivity of the entropy to the state ρ_{12} , see [11, 3].

(ii) The lower bound is a special case of the bound in theorem 3. As Φ_1 and Φ_2 are bistochastic

$$\Phi_1(\rho_*) = \Phi_2(\rho_*) = \rho_* \quad (67)$$

and the contribution of the terms within the square brackets vanishes. \square

Corollary 1. *Let Φ be a bistochastic map, then*

$$\mathbb{S}(\Phi^{on}) \leq n\mathbb{S}(\Phi). \quad (68)$$

It should be remarked that (68) is only meaningful when $S(\Phi) \ll \ln N$ and n is not too large as the entropy of a map is anyway bounded by $2 \ln N$.

In a similar way one may analyse properties of a concatenation of three consecutive operations. Motivated by the inequality (51) for the entropy of the products of three bistochastic matrices, we formulate its quantum mechanical counterpart.

Theorem 2 (Strong dynamical subadditivity for bistochastic quantum operations). *Let Φ_1, Φ_2 and Φ_3 be bistochastic quantum operations then their entropies satisfy the inequality*

$$S(\Phi_3 \circ \Phi_2 \circ \Phi_1) + S(\Phi_2) \leq S(\Phi_3 \circ \Phi_2) + S(\Phi_2 \circ \Phi_1). \quad (69)$$

This is equivalent to

$$S(\sigma_3 \odot \sigma_2 \odot \sigma_1) + S(\sigma_2) \leq S(\sigma_3 \odot \sigma_2) + S(\sigma_2 \odot \sigma_1), \quad (70)$$

where $\sigma_1, \sigma_2, \sigma_3 \in \mathcal{D}_{N^2}^{\text{II}}$.

Proof. Consider three bistochastic quantum operations Φ_1, Φ_2 and Φ_3 acting in sequence on the maximally mixed state $\rho_* \in \mathcal{M}_N$. We repeat the construction of the Lindblad operator as in (59) but now for three ancillas E_1, E_2 and E_3 and obtain a density matrix ω_{321} acting on $\mathcal{H}_N \otimes E_3 \otimes E_2 \otimes E_1$. The restrictions of ω_{321} to some of the ancillas will be denoted as before by ρ 's. We now write the strong subadditivity of quantum entropy for the system $E_3 E_2 E_1$ [9]

$$S(\rho_{321}) + S(\rho_2) \leq S(\rho_{21}) + S(\rho_{32}). \quad (71)$$

This yields, using the bistochasticity of Φ_3 and Φ_2 ,

$$S(\rho_2) = S(\Phi_2) \quad (72)$$

$$S(\rho_{21}) = S(\Phi_2 \circ \Phi_1) \quad (73)$$

$$S(\rho_{32}) = S(\Phi_3 \circ \Phi_2) \quad (74)$$

$$S(\rho_{321}) = S(\Phi_3 \circ \Phi_2 \circ \Phi_1), \quad (75)$$

which ends the proof. □

4.2. Generalization of dynamical subadditivity for stochastic maps

Results obtained in the previous section for bistochastic maps can be generalized for the case of arbitrary stochastic maps.

Theorem 3 (Dynamical subadditivity for quantum operations). *Let Φ_1 and Φ_2 be quantum operations then their entropies satisfy the inequalities*

$$S(\Phi_1) + \Delta_1 \leq S(\Phi_2 \circ \Phi_1) \leq S(\Phi_1) + S(\Phi_2) + \Delta_2 \quad (76)$$

where

$$\Delta_1 = S(\Phi_2 \circ \Phi_1(\rho_*)) - S(\Phi_1(\rho_*)), \quad (77)$$

$$\Delta_2 = S(\hat{\sigma}(\Phi_2, \Phi_1(\rho_*))) - S(\Phi_2). \quad (78)$$

Proof. We repeat the proof of theorem 1 for the composition $\Phi_2 \circ \Phi_1$ up to the construction of the state ω on $\mathcal{H}_N \otimes E_2 \otimes E_1$, see (59). If the first operation is not bistochastic then the restriction (66) has the form

$$\rho_2 = \text{Tr}_{\mathcal{H}_N \otimes E_1} \omega_{21} = \text{Tr}_{\mathcal{H}_N} \sum_{\alpha_2, \beta_2=1}^N B_{\omega_2} \Phi_1(\rho_*) B_{\beta_2}^\dagger \otimes |\alpha_2\rangle\langle\beta_2| = \hat{\sigma}(\Phi_2, \Phi_1(\rho_*)). \quad (79)$$

The restrictions ρ_{12} and ρ_1 are the same as before. The entropy of $\hat{\sigma}$ forms the first term of Δ_2 so subadditivity of the entropy of ρ_{12} implies the upper bound of (76).

To prove the lower bound consider (63). As an initial state let us take $\rho_* = \frac{1}{N} \mathbb{1}$ and purify it by adding an additional party R of dimension N . This yields a pure state on $R \otimes \mathcal{H}_N \otimes E_2 \otimes E_1$ defined by the normalized vector

$$\frac{1}{N} \sum_{i=1}^N \sum_{\alpha_1, \alpha_2=1}^{N^2} |i\rangle \otimes (B_{\alpha_2} A_{\alpha_1} |i\rangle) \otimes |\alpha_2\rangle \otimes |\alpha_1\rangle. \quad (80)$$

We now use the strong subadditivity of the entropy denoting by ρ_R the restriction of this pure state to the party R with similar notations for restrictions to other parties

$$S(\rho_{RE_2E_1}) + S(\rho_{E_1}) \leq S(\rho_{RE_1}) + S(\rho_{E_2E_1}). \quad (81)$$

Computing all the terms appearing in (81),

$$S(\rho_{E_1}) = S(\Phi_1) \quad (82)$$

$$S(\rho_{E_2E_1}) = S(\Phi_2 \circ \Phi_1) \quad (83)$$

$$S(\rho_{RE_1}) = S(\Phi_1(\rho_*)) \quad (84)$$

$$S(\rho_{RE_2E_1}) = S(\Phi_2 \circ \Phi_1(\rho_*)) \quad (85)$$

we arrive at the lower bound (76). □

The above inequalities formulated in the language of quantum maps obviously hold for the composition of states which belong to the set (43) and its subset (44).

The above results may be linked to properties of *coherent information* defined by Schumacher and Nielsen [14] as a function of exchange entropy $S(\hat{\sigma}(\Phi, \rho))$:

$$I(\Phi, \rho) = S(\Phi(\rho)) - S(\hat{\sigma}(\Phi, \rho)). \quad (86)$$

Denoting by I_1 the coherent information for the first operation, $I_1 = S(\Phi_1(\rho)) - S(\hat{\sigma}(\Phi_1, \rho))$, and by I_{21} the analogous quantity for the concatenation $I_{21} = S(\Phi_2(\Phi_1(\rho))) - S(\hat{\sigma}(\Phi_2 \circ \Phi_1, \rho))$, these authors proved [14] the *quantum data processing inequality*:

$$I_{21} \leq I_1. \quad (87)$$

This result implies the lower bound of (76).

4.3. Entropy of a product of stochastic matrices

Restricting our attention to the case of diagonal dynamical matrices we may analyse classical analogues of the above results. In this case an application of a quantum map corresponds to action of a stochastic matrix T on a classical probability vector P .

Theorem 3 implies that in the classical case for an arbitrary stochastic matrices T_1 and T_2 the following bounds hold,

$$H(T_1) + [H(T_2 T_1(P_*)) - H(T_1(P_*))] \leq H(T_2 T_1) \leq H(T_1) + H_{T_1 P_*}(T_2) + H(T_1 P_*), \quad (88)$$

where $P_* = \frac{1}{N}(1, \dots, 1)$.

However, it is possible to obtain a stronger upper bound by using properties of strong subadditivity of entropy. Inequality (71) is equivalent to an inequality for the exchange entropy,

$$S(\hat{\sigma}(\Phi_3 \circ \Phi_2 \circ \Phi_1, \rho_*)) + S(\hat{\sigma}(\Phi_2, \Phi_1(\rho_*))) \leq S(\hat{\sigma}(\Phi_2 \circ \Phi_1, \rho_*)) + S(\hat{\sigma}(\Phi_3 \circ \Phi_2, \Phi_1(\rho_*))). \quad (89)$$

By restriction to the classical case we obtain an inequality for three stochastic matrices $T_1 = T_1(\Phi_1)$, $T_a = T_a(\Phi_2)$ and $T_2 = T_2(\Phi_3)$ (note the notation chosen, which is convenient to state the final result). This inequality is similar to the strong subadditivity, but different weights for entropies are used. By substitution $T_a = \mathbb{1}$ we arrive with a result analogous to (76).

Theorem 4. *Let T_1 and T_2 be arbitrary stochastic matrices. Then the entropy of their product is bounded by*

$$H(T_1) + \delta_1 \leq H(T_2 T_1) \leq H(T_2) + H(T_1) + \delta_2 \quad (90)$$

where

$$\delta_1 = H(T_2 T_1(P_*)) - H(T_1(P_*)), \quad (91)$$

$$\delta_2 = H_{T_1 P_*}(T_2) - H(T_2). \quad (92)$$

This classical version of (76) valid for arbitrary stochastic matrices T_1 and T_2 can be considered as a direct generalization of the result of Słomczyński [15] obtained for bistochastic matrices. If both matrices T_1 and T_2 are bistochastic a complementary lower bound for the entropy of their product holds $H(T_2) \leq H(T_2 T_1)$, so in this case (50) can be rewritten in the stronger symmetric form [19]

$$\max(\{H(T_1), H(T_2)\}) \leq \min(\{H(T_1 T_2), H(T_2 T_1)\}). \quad (93)$$

5. Concluding remarks

In this paper we introduced the composition \odot between states on a bipartite system and derived some basic properties. This composition reflects the concatenation of quantum operations under the Jamiołkowski isomorphism. Next, we introduced a simple notion of entropy of a quantum map in order to quantify its randomizing properties. We proved the property of subadditivity for a composition of arbitrary bistochastic maps and found its generalization for the case of stochastic quantum maps. The connection between maps and states allows us to formulate these properties purely in terms of states and the \odot composition. A restriction to the classical setting leads to a generalization of recently obtained bounds on the entropy of a product of two bistochastic matrices [15] to the case of a product of arbitrary stochastic matrices.

Recently [17], the concatenation of quantum maps has been investigated from the point of view of divisibility. The authors consider the determinant of a super-operator instead of the entropy and show that it is contractive with respect to composition.

A more detailed understanding of the randomizing properties of a quantum map should be provided by constructing a Markov-like process. A possible track is to generate a state on a spin half-chain in the spirit of finitely correlated states [6] and to consider the associated dynamical entropy. This is a subject of future research.

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References

- [1] Alicki R and Fannes M 2001 *Quantum Dynamical Systems* (Oxford: Clarendon)
- [2] Araki H and Lieb E 1970 Entropy inequalities *Commun. Math. Phys.* **18** 160–70
- [3] Bengtsson I and Życzkowski K 2006 *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge: Cambridge University Press)
- [4] Choi M-D 1975 Completely positive linear maps on complex matrices *Linear Algebra Appl.* **10** 285–90
- [5] Dierckx B, Fannes M and Pogorzelska M 2007 Quasi-free states and maps in information theory *Preprint quant-ph/0709.1061*
- [6] Fannes M, Nachtergaele B and Werner R F 1992 Finitely correlated states on quantum spin chains *Commun. Math. Phys.* **144** 443–90
- [7] Jamiołkowski A 1972 Linear transformations which preserve trace and positive semi-definiteness of operators *Rev. Mod. Phys.* **3** 275–8
- [8] Kraus K 1971 General state changes in quantum theory *Ann. Phys.* **64** 311–35
- [9] Lieb E H and Ruskai M-B 1980 Proof of the strong subadditivity of quantum mechanical entropy *J. Math. Phys.* **14** 1938–41
- [10] Lindblad G 1991 Quantum entropy and quantum measurements *Quantum Aspects of Optical Communication (Lecture Notes in Physics vol 378)* ed C Bendjaballah *et al* (Berlin: Springer) p 36
- [11] Ohya M and Petz D 1993 *Quantum Entropy and Its Use* (Berlin: Springer)
- [12] Roga W 2007 Entropie złożonych operacji kwantowych *Master Thesis* (in Polish), Jagiellonian University, Cracow see <http://chaos.if.uj.edu.pl/~karol/prace/Roga07.pdf>
- [13] Schumacher B 1996 Sending entanglement through noisy quantum channels *Phys. Rev. A* **54** 2614–28
- [14] Schumacher B and Nielsen M A 1996 Quantum data processing and error correction *Phys. Rev. A* **54** 2629–35
- [15] Słomczyński W 2002 Subadditivity of entropy for stochastic matrices *Open Syst. Inf. Dyn.* **9** 201–6
- [16] Sudarshan E C G, Mathews P M and Rau J 1961 Stochastic dynamics of quantum mechanical systems *Phys. Rev.* **121** 920–4
- [17] Wolf M M and Cirac J I 2006 Dividing quantum channels *Preprint math-ph/0611057*
- [18] Życzkowski K and Bengtsson I 2004 On duality between quantum states and quantum maps *Open Syst. Inf. Dyn.* **11** 3–42
- [19] Życzkowski K, Kuś M, Słomczyński W and Sommers H-J 2003 Random unistochastic matrices *J. Phys. A: Math. Gen.* **36** 3425–50